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Stable Complete Surfaces with Constant Mean Curvature

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Abstract. Let $x : M^2 \longrightarrow N^3$ be a stable immersion with constant mean curvature H of a complete orientable surface M^2 into a complete oriented three dimensional Riemannian manifold N^3. In this paper we prove that, if M^2 is compact and $H^2 > -\frac{1}{2}\inf_M \mathrm{Ricc}_N$, then M^2 has genus $g \leq 3$, here Ricc_N is the Ricci curvature of N^3. We also prove that, if M^2 is complete non compact and N^2 has bounded geometry, the area of M^2 is infinite in the metric induced by x. In this case, if $H^2 \geq -\frac{1}{2}\inf_M \mathrm{Ricc}_N$ then x is umbilic and the equality holds.

1. Introduction

The goal of this paper is to present some results on the stability of immersions $x : M^2 \longrightarrow N^3$ with constant mean curvature H.

We first consider the case where M^2 is compact, orientable, and prove the following result.

"Let M^2 have genus g. If $x : M^2 \longrightarrow N^3$ is stable and

$$H^2 > -\frac{1}{2}\inf_M \mathrm{Ricc}_N,$$

then $g \leq 3$. Here Ricc_N is the Ricci curvature of N".

This is related to Fischer-Colbrie and Schoen [**9**] where it was proved that if x is minimal and stable, and N^3 has nonnegative Ricci curvature, then $g \leq 1$. A similar result was also obtained by El Soufi and S. Ilias [**6**].

It was proved in [**2**] that if M^2 is a compact, stable surface with constant mean curvature H in the three dimensional simply-connected, complete Riemannian manifold Q_c^3 with constant sectional curvature c,

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then $M^2 \subset Q_c^3$ is a geodesic sphere and $(H^2+c)A = 4\pi$, where A is the area of M^2. It is a surprising fact that this equality becomes a sharp inequality in the case that the ambient space N^3 is arbitrary.

"Let M^2 be compact, orientable and assume that $x : M^2 \longrightarrow N^3$ is a stable surface with constant mean curvature H. Let A be the area of M^2 and $c = \frac{1}{2}\inf_M \mathrm{Ricc}_N$. Then

$$(H^2+c)A \leq 4\pi$$

and equality holds if and only if M^2 has genus $g = 0$, x is umbilic, and the sectional curvature $K_N \equiv c$ in M".

We next consider the case where M^2 is complete noncompact. For that case, we will need that the area of M^2 is infinite in the metric induced by $x : M^2 \longrightarrow N^3$. This will be seen to be the case if N^3 has *bounded geometry* (i.e., it has sectional curvature bounded above and injectivity radius bounded below). More generally, we will prove in an Appendix to this paper the following result.

"Let M^m be a complete, noncompact manifold and let $x : M^m \longrightarrow N^n$ be an immersion with mean curvature vector field bounded in norm. Assume that N^n has bounded geometry. Then the volume of M^m in the induced metric is infinite".

Now we can present our results on stability.

"Let M^2 be a complete, noncompact, orientable surface and let $x : M^2 \longrightarrow N^3$ be an immersion with constant mean curvature H. Assume that N^3 has bounded geometry and that $H^2 \geq -\frac{1}{3}\inf_M S$ where S is the scalar curvature of N^3. If M^2 has finite index, then $H^2 = -\frac{1}{3}\inf_M S$".

From this theorem we obtain that, *if H is nonzero and N^3 has nonnegative scalar curvature, then M^2 has finite index if and only if M^2 is compact.*

Our final theorem generalizes results of Fischer-Colbrie and R. Schoen [**9**] for stable minimal surfaces and of A.M. Silveira [**12**] for stable surfaces with constant mean curvature in Q_c^3.

"Let M^2 be a complete, noncompact, orientable surface and let

$x : M^2 \longrightarrow N^3$ be an immersion with constant mean curvature H. Assume that N^3 has bounded geometry and that $H^2 \geq -\frac{1}{2}\inf_M \mathrm{Ricc}_N$. If x is stable, then

$$H^2 = -\frac{1}{2}\inf_M \mathrm{Ricc}_N$$

and x is umbilic. Furthermore, M^2 is conformally equivalent to the plane or to a cylinder. If M^2 is a cylinder, then M^2 is flat"

The results of this paper are part of my Doctoral Dissertation at IMPA, as were announced in [**10**]. Personal problems prevented its publication at the proper time. I want to thank to M. do Carmo for his orientation.

2. Stability of Compact Surfaces with Constant Mean Curvature.

Let M^2 be a complete orientable surface and let N^3 be a three dimensional complete oriented Riemannian manifold. Let $x : M^2 \longrightarrow N^3$ be an isometric immersion with constant mean curvature H. We know that such surfaces are critical points of the area function for compactly supported variation that preserve volume. We say that the immersion is *stable* if the second variation of the area function is nonnegative, i.e.,

$$I(f) = \int_M [-f\Delta f - (\mathrm{Ricc}_N(e_3) + \|B\|^2)f^2]dA \geq 0 \tag{1}$$

for any piecewise smooth function $f : M \longrightarrow \mathbb{R}$ with compact support and with $\int_M f dA = 0$ (see [**2**]). Here Δ is the Laplacian in M, e_3 is a unit normal vector field, $\mathrm{Ricc}_N(e_3)$ is the Ricci curvature of N in the direction of e_3 and $\|B\|$ is the norm of the second fundamental form B.

Associated to the quadratic form I we have the operator

$$L = \Delta + \mathrm{Ricc}_N(e_3) + \|B\|^2,$$

called the *Jacobi operator*.

Let (M^2, ds^2) be a compact orientable surface of genus g. Then M^2 has a Riemman surface structure compatible with the metric ds^2.

In the study of stability of a compact surface with constant mean curvature, it is fundamental to have some way of constructing smooth functions $f : M \longrightarrow \mathbb{R}$ with mean value zero, i.e. $\int_M f dA = 0$. For this, we will use in this section the following result.

Lemma 1. **(see [5], Lemma (2.1).)** *Let $S^2 \subset \mathbb{R}^3$ be the round sphere of radius one and let $\psi : M^2 \longrightarrow S^2$ be a nonconstant meromorphic function. Then there exists a conformal transformation $\phi : S^2 \longrightarrow S^2$ such that*

$$\int_M \overline{\psi}_i dA = 0$$

for $i = 1, 2, 3$, where $\overline{\psi} = \phi \circ \psi = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3) \in \mathbb{R}^3$.

Let e_1, e_2, e_3 be a positively oriented orthonormal frame defined locally on M with e_1 and e_2 tangent to M, and e_3 the unit normal vector field. Let $b_{ij} = \langle \overline{\nabla}_{e_i} e_3, e_j \rangle$, $1 \leq i, j \leq 2$, be the coeficients of the second fundamental form B, where $\overline{\nabla}$ is the Riemannian connection of N.

By the Gauss formula, we have

$$K = K_{12} + b_{11}b_{22} - b_{12}^2$$

where K is the Gaussian curvature of M and K_{ij} is the sectional curvature of N for the plane determined by e_i and e_j.

Therefore,

$$2H^2 - K + K_{12} = \frac{(b_{11} + b_{22})^2}{2} - b_{11}b_{22} + {b_{12}}^2 = \frac{\|B\|^2}{2} \qquad (2)$$

and

$$H^2 - K + K_{12} = \frac{(b_{11} - b_{22})^2}{4} + b_{12}^2 \geq 0. \qquad (3)$$

We now prove the results about stability of compact surfaces with constant mean curvature mentioned in the introduction.

Theorem 2. *Let (M^2, ds^2) be a compact orientable surface of genus g, and let $x : M^2 \longrightarrow N^3$ be an isometric immersion with constant mean curvature H. If x is stable and $H^2 > -\frac{1}{2} \inf_M \mathrm{Ricc}_N$, where Ricc_N is the Ricci curvature of N, then $g \leq 3$.*

Proof. Suppose that $g \geq 4$. Then there exists a nonconstant meromorphic function $\psi : M^2 \longrightarrow S^2$ of degree less or equal than $(3g+1)/4$ (see [7], pg. 118). So, by Lemma 1, there exists a nonconstant meromorphic function $\overline{\psi} : M^2 \longrightarrow S^2$ of degree less or equal to $(3g+1)/4$ such that

$$\int_M \overline{\psi}_i dA = 0,$$

for $i = 1, 2, 3$, where $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3)$. Since x is stable, we have

$$\int_M |grad\,\overline{\psi}_i|^2 dA \geq \int_M (-2K + 4H^2 + 2K_{12} + \mathrm{Ricc}_N(e_3))\overline{\psi}_i^{\,2}\, dA,$$

$i = 1, 2, 3$, where *grad* denotes the gradient in M.

By summing up in i, $i = 1, 2, 3$ and by using that $|\overline{\psi}|^2 = \sum_{i=1}^{3} \overline{\psi}_i^{\,2} = 1$ we obtain that

$$\int_M |\nabla\overline{\psi}|^2 dA \geq \int_M (-2K + 4H^2 + 2K_{12} + \mathrm{Ricc}_N(e_3)) dA, \tag{4}$$

where $|\nabla\overline{\psi}|^2 = \sum_{i=1}^{3} |grad\,\overline{\psi}_i|^2$. By using that $\int_M K dA = 4\pi(1-g)$, that

$$\int_M |\nabla\overline{\psi}|^2 dA = 8\pi(degree\,(\overline{\psi})),$$

and that

$$2K_{12} + \mathrm{Ricc}_N(e_3) = \mathrm{Ricc}_N(e_1) + \mathrm{Ricc}_N(e_2) \geq -4c,$$

we obtain

$$8\pi(degree\,(\overline{\psi})) + 8\pi(1-g) \geq 4(H^2 - c)A, \tag{5}$$

where A is the area of M^2 and $c = -\frac{1}{2}\inf_M \mathrm{Ricc}_N$.

If $g = 4$, $degree\,(\overline{\psi}) \leq 13/4$. So $degree\,(\overline{\psi}) \leq 3$, and by (5), $4(H^2 - c)A \leq 0$, which is a contradiction. Now suppose that $g \geq 5$. By (5),

$$4(H^2 - c)A \leq 8\pi\frac{3g+1}{4} + 8\pi(1-g) \leq 2\pi(-g+5) \leq 0,$$

which is again a contradiction. It follows that $g \leq 3$ and the theorem is proved. □

Corollary 3. *There is no compact orientable stable surface of genus $g \geq 4$ with constant mean curvature in a three-dimensional manifold with positive Ricci curvature.*

Remark 4 . Corollary 3 also holds if the ambient space has nonnegative Ricci curvature and the compact surface has *nonzero* constant mean curvature.

Remark 5 . M. Ross proved in [**14**] that the classical Schwarz $\mathcal{P}$-minimal surface of genus three in the flat three torus is a stable constant mean curvature surface. He also mentions ([**14**], p. 193) that the constant mean curvature companions of the Schwarz $\mathcal{P}$-minimal surface that are close enough to it are stable. Thus, the result of Theorem 2 is sharp.

Remark 6 . It follows from the above Remark that it is unlikely that an explicit description for stable surface can be found even in the flat 3-torus. However, M. Ritoré and A. Ros proved in [**13**] that if M^2 is a stable compact orientable surface with nonzero constant mean curvature in the real projective space $\mathbf{RP}^3$, then either M^2 is a geodesic sphere ($g = 0$) or is an embedded flat torus ($g = 1$) of radius r, with $\pi/6 \leq r \leq \pi/3$, about a geodesic. If $H = 0$, they also proved that M^2 is a two fold covering of a real projective plane ($g = 0$).

Theorem 7. *Let M^2 be a compact, orientable surface, and let $x : M^2 \to N^3$ be an isometric immersion with constant mean curvature H. If x is stable, then*

$$(H^2 + c)A \leq 4\pi,$$

where $c = \frac{1}{2}\inf_M \mathrm{Ricc}_N$ and $A = \mathrm{area}(M)$. Furthermore, the equality holds if and only if M^2 has genus zero, x is umbilic and $K_N \equiv c$ in M^2.

Proof. Let g be the genus of M. From the Riemann-Roch Theorem, there exists a finite number of points $p_1, \ldots, p_k$, called Weierstrass points of M, such that, if $p \in M - \{p_1, \ldots, p_k\}$, then there exists a meromorphic function $\psi : M \longrightarrow S^2$ such that ψ is holomorphic in $M - \{p\}$, and p is a pole of order $g + 1$ of ψ. Then $degree\psi = g + 1$. By Lemma 1, we obtain a conformal transformation $\phi : S^2 \longrightarrow S^2$ such that

$$\int_M \overline{\psi} dA = 0$$

where $\overline{\psi} = \phi \circ \psi$. Then, by (4),

$$\int_M |\nabla\overline{\psi}|^2 dA \geq \int_M (-2K + 4H^2 + 2K_{12} + \mathrm{Ricc}_N(e_3))dA,$$

since x is stable. Using the facts that

$$\int_M |\nabla\overline{\psi}|^2 dA = 8\pi(g+1), \qquad \int_M K dA = 4\pi(1-g),$$

and $\mathrm{Ricc}_N \geq 2c$ in M, we obtain

$$8\pi(g+1) \geq -8\pi(1-g) + 4(H^2+c)A.$$

Thus $(H^2+c)A \leq 4\pi$. Now we suppose that $(H^2+c)A = 4\pi$. Then $K_N \equiv c$ in M, and equality holds in (4). Therefore,

$$\Delta\overline{\psi} + (-2K + 4H^2 + 4c)\overline{\psi} = 0 \text{ in } M. \tag{6}$$

In the other hand, since $\overline{\psi}$ is a meromorphic function we have

$$\Delta\overline{\psi} + |\nabla\overline{\psi}|^2\overline{\psi} = 0 \text{ in } M. \tag{7}$$

Equalities (6) and (7) imply that

$$|\nabla\overline{\psi}|^2(q) = (-2K + 4H^2 + 4c)(q),$$

for any point $q \in M$.

If $g > 0$, $|\nabla\overline{\psi}|^2(p) = 0$, since p is a pole of order $(g+1) \geq 2$. Then $K(p) = 2(H^2+c)$. Because this equality holds for every point $p \in M - \{p_1, \ldots, p_k\}$ we obtain that $K \equiv 2(H^2+c)$ in M. Since $H^2 - K + K_{12} \geq 0$, we have that $H^2 + c \leq 0$, wich is a contradiction. Then $g = 0$.

Moreover, since $\|B\|^2 + \mathrm{Ricc}_N(e_3) = -K + 3H^2 + 3c + (H^2 - K + K_{12})$, and the equality holds in (4), we have

$$\int_M [|\nabla\overline{\psi}|^2 + K - 3H^2 - 3c - (H^2 - K + K_{12})]dA = 0.$$

By using that $\int_M \|\nabla\overline{\psi}\|^2 dA = 8\pi$, that $\int_M K dA = 4\pi$, and that $(H^2+c)A = 4\pi$, we conclude that $\int_M (H^2 - K + K_{12})dA = 0$. Then, by (3), $H^2 - K + K_{12} \equiv 0$ in M, i.e., x is umbilic.

Now suppose that M^2 is an umbilic sphere in N^3 and $K_N \equiv c$ in M. Then

$$4\pi = \int_M K dA = \int_M (H^2+c)dA = (H^2+c)A,$$

and this completes the proof of the theorem. □

3. Stability of Complete Noncompact Surfaces with Constant Mean Curvature.

Let M^2 be a complete orientable surface and let $T = \Delta + q$ be an operator in M, where $q : M \longrightarrow \mathbb{R}$ is a smooth function. Let D be a relatively compact domain in M, with smooth boundary. The *index* of T in D, denoted by $\mathrm{Ind}_T(D)$, is the number of negative eigenvalues of T with Dirichlet boundary condition. We define the *index* of T in M by

$$\mathrm{Ind}_T(M) = \sup_{D \subset M} \mathrm{Ind}_T(D),$$

where D is any relatively compact domain in M.

When T is the Jacobi operator L, the index of T in M is called the *index* of M, and is denoted by $\mathrm{Ind}(M)$. One can prove (see [**13**]) that if the immersion $x : M^2 \longrightarrow N^3$ is stable with constant mean curvature then $\mathrm{Ind}(M)$ is at most one. For minimal surfaces, we know that the condition $I(f) \geq 0$ for all compactly supported function f is a necessary and sufficient condition for stability. Thus, a minimal immersion is stable if and only if $\mathrm{Ind}(M) = 0$.

To prove our results we will need the following theorems:

Theorem 1. (**A.M. da Silveira [15], pg.630.**) *Let $(M^2,\ ds^2)$ be a complete surface and let $T = \Delta - K + q$ be an operator as above, where K is the Gaussian curvature of M. Assume that the operator has finite index and that the function q is nonnegative. Then M^2 is conformally equivalent to a compact Riemannian surface minus a finite number of points. Furthermore,*

$$\int_M q\,dA < \infty,$$

where dA is the area element in the metric ds^2.

Theorem 2. (**A.M. da Silveira [15], pg. 630.**) *Let (M^2, ds^2) be a complete surface conformally equivalent to a compact Riemann minus a finite number of points, and let $T = \Delta + q$. Assume that q is nonnegative, $q \neq 0$, and that the area of M is infinite. Then there exists a piecewise* *smooth function* $f : M \longrightarrow \mathbb{R}$ *with compact support such that*

$$\int_M -fT(f)dA < 0, \text{ and } \int_M f dA = 0.$$

Now we will give the proof of the theorems mentioned in the introduction.

Theorem 3. *Let* M^2 *be a complete, noncompact, orientable surface, and let* $x : M^2 \longrightarrow N^3$ *be an isometric immersion with constant mean curvature* H. *Assume that* N^3 *has bounded geometry and* $H^2 \geq -\frac{1}{3}\inf_M S$, *where* S *is the scalar curvature of* N^3. *If* M^2 *has finite index, then* $H^2 = -\frac{1}{3}\inf_M S$.

Proof. By (2) Jacobi's operator may be rewritten in the form

$$L = \Delta - K + q,$$

where $q = 4H^2 - K + K_{12} + S$, and S is the scalar curvature of N given by $S = K_{12} + K_{13} + K_{23} = K_{12} + \mathrm{Ricc}_N(e_3)$. Since $3H^2 + \inf_M S \geq 0$ and $H^2 - K + K_{12} \geq 0$, we obtain that $q \geq 3H^2 + \inf_M S \geq 0$. Because the operator L has finite index, we obtain by Theorem 1 that

$$\int_M (3H^2 + \inf_M S)dA \leq \int_M q dA < \infty.$$

On the other hand, since N has bounded geometry, we have that M has infinite area (see Theorem 1 in the Appendix). So $H^2 = -\frac{1}{3}\inf_M S$, and this completes the proof. □

Corollary 4. *Let* M^2 *be a complete orientable surface, and let* $x : M^2 \longrightarrow N^3$ *be an isometric immersion with constant nonzero mean curvature* H. *Assume that* N^3 *has bounded geometry and nonnegative scalar curvature. Then* M^2 *has finite index if and only if* M^2 *is compact.*

Proof. Suppose that M is compact. Since L is an elliptic operator, the index of L in M is finite.

If M has finite index and it is noncompact, we obtain, by Theorem 3, that $H = 0$. So M is compact. □

It should be remarked that when the immersion is minimal the above situation changes. In fact, it has been proved in [**8**] and [**11**] that if M^2 is a minimal surface in $\mathbb{R}^3$, then $\mathrm{Ind}(M) < \infty$ if and only if $\int_M (-K)dA <$

∞. Thus, there exist several examples of noncompact minimal surfaces in $\mathbb{R}^3$, with finite index.

Corollary 5. *If M^2 is a complete surface in the hyperbolic three-space of sectional curvature -1, with constant mean curvature $H > 1$, then $\operatorname{Ind}(M) < \infty$ if and only if M is compact.*

If $H = 1$, M. do Carmo and A.M. Silveira proved in [4] that $\operatorname{Ind}(M) < \infty$ if and only if $\int_M (-K)dA < \infty$. Thus there are examples of noncompact surfaces with $H = 1$ in the hyperbolic space with finite index.

Corollary 6. *There is no complete, noncompact surface with constant mean curvature and finite index in a three-dimensional compact Riemannian manifold with positive scalar curvature.*

Theorem 7. *Let M^2 be a complete, noncompact surface with constant mean curvature and let $x : M^2 \longrightarrow N^3$ be an isometric immersion with constant mean curvature H. Assume that N^3 has bounded geometry. If $H^2 \geq -\frac{1}{2}\operatorname{Ricc}_N$, where Ricc_N is the Ricci curvature of N, and x is stable, then $H^2 = -\frac{1}{2}\inf_M \operatorname{Ricc}_N = -\frac{1}{2}\operatorname{Ricc}_N(e_3)$ and x is umbilic. Furthermore, M^2 is conformally equivalent to the complex plane or the cylinder. If M^2 is a cylinder, then M^2 is flat.*

Proof. Since x is stable and $H^2 \geq -\frac{1}{2}\operatorname{Ricc}_N \geq -\frac{1}{3}\inf_M S$, we have, by Theorem 3 that $H^2 = -\frac{1}{3}\inf_M S = -\frac{1}{2}\inf_M \operatorname{Ricc}_N$. From Theorem 1 we also have that M^2 is conformally equivalent to a compact Riemann surface minus a finite number of points.

The Jacobi operator L can be rewritten in the form $L = \Delta + q$, where $q = 2(H^2 - K + K_{12}) + 2H^2 + \operatorname{Ricc}_N(e_3)$. Since x is stable, $H^2 - K + K_{12} \geq 0$ and $2H^2 + \operatorname{Ricc}_N(e_3) \geq 0$, we obtain by Theorem 2 that $H^2 - K + K_{12} \equiv 0$ and $H^2 \equiv -\frac{1}{2}\operatorname{Ricc}_N(e_3)$ on M^2. So, the immersion x is umbilic.

On the other hand, since

$$-3H^2 \leq S = K_{12} + \operatorname{Ricc}_N(e_3) = K_{12} - 2H^2,$$

we have that $K = K_{12} + H^2 \geq 0$. It follows that the operator $\Delta - K$ is positive semidefinite. So, there is a smooth positive function $f : M \longrightarrow \mathbb{R}$ that satisfies $\Delta f - Kf = 0$ on M^2 (see Theorem 1 in [**9**]). Therefore, by Theorem 2 in [**9**], we know that the universal covering of M^2 is conformally equivalent to the complex plane. So, M^2 is conformally equivalent to the complex plane or the cylinder. If M^2 is conformally equivalent to the cylinder we have by the Cohn-Vossen inequality, that $\int_M KdA \leq 0$. Since $K \geq 0$, it follows that $K = 0$. This completes the proof. □

Remark 8 . If N^3 has nonnegative Ricci curvature, the theorem above extends to surfaces with constant mean curvature the result obtained by F. Colbrie and R. Schoen in [**9**].

Remark 9 . If $N^3 = \mathbb{Q}^3(c)$, $c = -1, 0, 1$, the theorem above yields the results proved by A.M da Silveira in [**15**].

Appendix

Let N^n be a complete Riemannian manifold of dimension n. We denote by $i_N(p)$ the injectivity radius of N in p and by K_N the sectional curvature of N. As mentioned in the introduction, a manifold N^n has bounded geometry if there exist positive real numbers δ and λ such that $K_N \leq \delta^2$ and $i_N \geq \lambda$ on N^n.

The purpose of this section is to prove the following result.

Theorem 1. *Let M^m be a complete, noncompact manifold, and let $x : M^m \to N^n$ be an isometric immersion with mean curvature vector field bounded in norm. If N^n has bounded geometry, the volume of M^m is infinite.*

Remark 2 . In the case that N has nonpositive sectional curvature, the result above has been proved by D. Hoffman and R. Schoen (personal communication).

In the proof of Theorem 1 we will use the following result which gives us a bound from below to the volume of a geodesic ball in M. We denote by $B_\mu(p)$ the geodesic ball in M^m with radius μ and center p, and by w_m the volume of the unit ball in $\mathbb{R}^m$.

Theorem 3. *Let $x : M^m \longrightarrow N^n$ be an isometric immersion with mean curvature vector field H bounded in norm. Assume that N^n has sectional* *curvature $K_N \leq \delta^2$, where δ is a positive real number. Then*

$$\mathrm{vol}_m(B_\mu(p)) \geq \delta^{-m} w_m (\sin \mu\delta)^m e^{-H_0\mu}, \tag{1}$$

where $\mu \leq \min\{\frac{\pi}{2\delta}, i_N(p)\}$ and $|H| \leq H_0$.

Remark 4 . When N^n has nonpositive sectional curvature we obtain, by letting δ tend to zero,

$$\mathrm{vol}_m(B_\mu(p)) \geq w_m \mu^m e^{-H_0\mu},$$

where $\mu \leq i_N(p)$.

For the proof of theorem 3 we will use the two lemmas below. First we give the following definitions.

Let $x : M^m \longrightarrow N^n$ be an isometric immersion. Given a vector field $V : M \longrightarrow TN$, its *gradient* $\overline{\nabla} V : TM \longrightarrow TN$ is the map $w \mapsto \overline{\nabla}_w V$, where $\overline{\nabla}$ is the covariant differentiation on N. The *divergence* of V on M, denoted by $div_M V$, is the trace of $\overline{\nabla} V$ on TM. If $e_1, \ldots, e_m$ is an orthonormal frame of T_pM, then

$$div_M V(p) = \sum_{i=1}^{m} \langle \overline{\nabla}_{e_i} V, e_i \rangle (p).$$

Lemma 5. **(see [12], pg. 719.)** *If $V : M \longrightarrow TN$ is a vector field on M, then*

$$div_M V^T = div_M V + \langle V, H \rangle, \tag{2}$$

where V^T denote the projection of V onto TM.

Lemma 6. **(see [12], pg. 721.)** *Let $x : M^m \longrightarrow N^n$ be an isometric immersion, $p_0 \in M^m$ and $r(\cdot) = d_N(\cdot, p_0)$, where d_N is the geodesic distance in N^n. Let $V = r\, grad\, r$ be the radial vector field centered at p_0, where grad is the gradient in N^n. If N^n has sectional curvature $K_N \leq \delta^2$, $\delta > 0$, then*

$$div_M V(p) \geq m\delta r(p) \cot(\delta r(p)),$$

for any point $p \in M$ such that $r(p) < \min\{\pi/\delta,\ i_N(p_0)\}$.

Remark. In the case that N^n has nonpositive sectional curvature, we have

$$div_M V(p) \geq m,$$

for any point $p \in M$ such that $r(p) < i_N(p_0)$. For this, we let δ tend to zero and observe that $s \cdot \cot(s) \longrightarrow 1$ as s tend to zero.

Proof of Theorem 3. Let p_0 be a point in M, $r(p) = d_N(p, p_0)$ and $V(p) = r(p)\, grad\, r(p)$. Let ρ be the function r restricted to M. Since $grad_M \rho = (grad r)^T$, we obtain by (2) that

$$\begin{aligned} \Delta_M \rho^2(p) &= 2div_M(\rho(grad\, r)^T)(p) \\ &= 2div_M V^T(p) = 2(div_M V + \langle V, H\rangle)(p) \\ &\geq 2(m\delta\rho \cot(\delta\rho) - \rho H_0)(p), \end{aligned}$$

for any point $p \in B_\mu(p_0)$, where Δ_M is the Laplacian on M and $\mu \leq \min\{\pi/\delta,\, i_N(p_0)\}$. By integrating the above expression,

$$\int_{B_\mu(p_0)} \Delta_M \rho^2 dA \geq 2m\delta C(\mu) - 2H_0 \int_{B_\mu(p_0)} \rho dA, \tag{3}$$

where $C(\mu) = \int_{B_\mu(p_0)} \rho \cot(\delta\rho) dA$. On the other hand, by using Stokes theorem, we obtain

$$\int_{B_\mu(p_0)} \Delta_M \rho^2 dA = \int_{\partial B_\mu(p_0)} grad_M \rho^2 \overrightarrow{\eta} dS,$$

where $\overrightarrow{\eta}$ is the exterior normal vector field to $B_\mu(p_0)$ on $\partial B_\mu(p_0)$. So, since $|grad_M \rho| \leq 1$ and $\rho(p) \leq d_M(p, p_0)$, we obtain

$$\int_{B_\mu(p_0)} \Delta_M \rho^2 dA \leq 2\mu A(\mu), \tag{4}$$

where $A(\mu) = \mathrm{vol}_{m-1}(\partial B_\mu(p_0))$.

Moreover, since $\rho(p) < \mu < \pi/2\delta$ and $p \in B_\mu(p_0)$,

$$\rho(p) = \rho(p)\cot(\delta\rho)(p)\tan(\delta\rho)(p) \leq \rho(p)\cot(\delta\rho)(p)\tan(\delta\mu). \tag{5}$$

By (3), (4) and (5),

$$\mu A(\mu) \geq (m\delta - H_0 \tan \mu\delta) C(\mu). \tag{6}$$

From the formula co-area formula (see [**3**] pag. 80), we obtain

$$\frac{dC}{d\mu} \geq \mu \cot(\delta\mu) A(\mu), \tag{7}$$

since the function $s \mapsto s \cdot \cot(s)$ is decreasing in $[0, \pi/2]$ and $\rho(p) \leq d_M(p, p_0)$. So by (6) and (7),

$$\frac{d}{d\mu}(C(\mu)(\sin \mu\delta)^{-m+1}) \geq (\sin \mu\delta)^{-m}(\delta \cos \mu\delta - H_0 \sin \mu\delta)C(\mu). \tag{8}$$

Therefore,

$$\frac{\frac{d}{d\mu}(C(\mu)(\sin \mu\delta)^{-m+1})}{C(\mu)(\sin \mu\delta)^{-m+1}} \geq \delta \cot \mu\delta - H_0.$$

By integrating the above expression from ϵ to μ, we obtain

$$\log\left(\frac{C(\mu)(\sin \mu\delta)^{-m}}{C(\epsilon)(\sin \epsilon\delta)^{-m}}\right) \geq -H_0(\mu - \epsilon).$$

It follows that

$$C(\mu)(\sin \mu\delta)^{-m} \geq C(\epsilon)(\sin \epsilon\delta)^{-m} e^{-H_0(\mu-\epsilon)}.$$

We now consider the function $\overline{\rho}(p) = d_M(p, p_0)$ and the function

$$\overline{C}(\epsilon) = \int_{B_\epsilon(p_0)} \overline{\rho} \cot(\overline{\rho}\delta) dA.$$

Since $C(\epsilon) \geq \overline{C}(\epsilon)$ and

$$\lim_{\epsilon \to 0+} \overline{C}(\epsilon)(\sin(\epsilon\delta))^{-m} = w_m \delta^{-m-1},$$

we obtain

$$C(\mu) \geq \delta^{-m-1} w_m \sin(\mu\delta)^m e^{-H_0\mu}.$$

On the other hand, since the function $s \mapsto s \cdot \cot(s)$ is decreasing and $s \cdot \cot(s) \longrightarrow 1$, as s tend to 0, we have

$$\mathrm{vol}_m(B_\mu(p_0)) \geq \delta C(\mu).$$

Then

$$\mathrm{vol}_m(B_\mu(p_0)) \geq \delta^{-m} w_m (\sin \mu\delta)^m e^{-H_0\mu},$$

and this completes the proof of Theorem 3. □

Proof of Theorem 1. Let $\lambda > 0$ be a real number such that $i_N(q) \geq \lambda$ for any point $q \in N^n$, and let $\mu_0 = \min\{\pi/2\delta, \lambda\}$. Then, by (1),

$$\mathrm{vol}_m(B_{\mu_0}(p)) \geq L > 0, \tag{9}$$

for any point $p \in M^m$, where

$$L = \delta^{-m} w_m (\sin \mu_0 \delta)^m e^{-H_0 \mu_0}.$$

Since M^m is complete and noncompact, it has a geodesic ray $\gamma : [0, \infty) \longrightarrow M^m$. Consider the sequence of points $p_j = \gamma(3j\mu_0)$, $j \geq 0$. Then, if $j \neq k$,

$$B_{\mu_0}(p_j) \cap B_{\mu_0}(p_k) = \varnothing.$$

So, by (9), for any positive integer k,

$$\mathrm{vol}_m(M) \geq \mathrm{vol}_m \left(\bigcup_{i=0}^{k} B_{\mu_0}(p_i) \right) \geq (k+1)L.$$

Since k is arbitrary, the volume of M^m is infinite, and this completes the proof of Theorem 1. □

Corollary 8. *Let M^m be a complete, noncompact Riemannian manifold, and let N^n be a compact Riemannian manifold. Let $x : M^m \longrightarrow N^n$ be an isometric immersion with mean curvature vector field bounded in norm. Then the volume of M^m is infinite.*

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